Sample GCHQ Mathematics Aptitude Test

The aptitude paper will consist of 2 sections. The first contains shorter questions, the solutions of which typically rely on the application of one or two ideas, while the second contains questions that will require a longer or more sophisticated train of thought. There will be plenty more questions on the paper than you could expect to answer in the time limit – our aim is primarily to find out those subject areas that you can do well. Therefore, you are advised to read the whole paper before starting and focus on those questions that you can approach confidently, as more credit is given for complete answers to a relatively small number of questions than for a large number of partial attempts. Results from this paper will be combined with those from the accompanying applications test.

Short Questions

1. Consider a 12 hour digital clock (one that takes values from 00:00 to 11:59). Look at it at a random point during a 12 hour period.

   What is the probability that you see at least one digit taking the value '1'?

   What is the probability that you see exactly one digit taking the value '1'?

2. Let a, b be positive integers and let p be a prime factor of $a^b - 1$.

   Show that either $gcd(p, a-1)$ or $gcd(b, p-1)$ must be greater that 1.

3. Alice and Bob are given a set of five biased coins. They both estimate the probability that each coin will show a head when flipped, and each coin is then flipped once. These are the estimates and values observed:

Coin	1	2	3	4	5
Alice's estimates	0.4	0.7	0.2	0.9	0.4
Bob's estimates	0.2	0.8	0.3	0.6	0.3
Observed	Heads	Heads	Tails	Tails	Heads

   Whom would you say is better at estimating the bias of the coins, and why?

4. Find an example of a function f from $[0,1]$ to $[0,1]$ with the following properties:

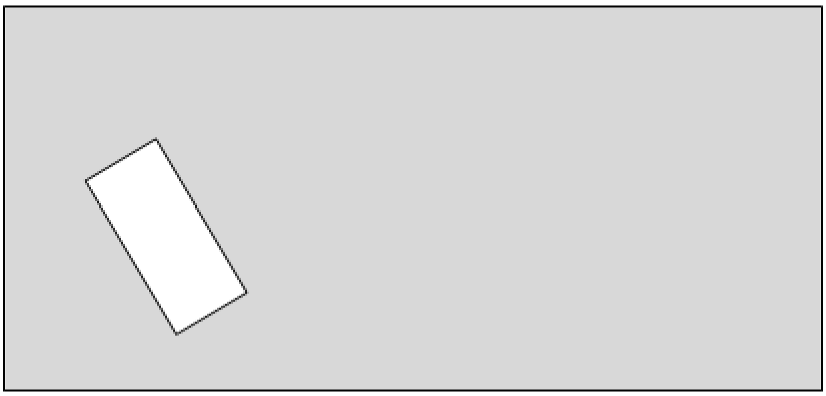
   i. f is continuous;

   ii. $f(0) = 0$;

   iii. $f(1) = 1$;

   iv. $f(x)$ is locally constant almost everywhere.

5. You have a large rectangular cake, and someone cuts out a smaller rectangular piece from the middle of the cake at a random size, angle and position in the cake (see the picture below). Without knowing the dimensions of either rectangle, using one straight (vertical) cut, how can you cut the cake into two pieces of equal area?



6. A random number generator produces independent random variates $x_0, x_1, x_2, \ldots$ drawn uniformly from $[0,1]$, stopping at the first x_T that is strictly less than x_0. Prove that the expected value of T is infinite. Suggest, with a brief explanation, a plausible value of $Pr(T = \infty)$ for a real-world (pseudo-)random number generator implemented on a computer.

7. Prove that there does not exist a four-digit square number n such that $n \equiv 1 \mod 101$.

8. The Fourier transform of a function f(t) is defined as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\omega t}\, dt.$$

   Express the Fourier transform of $f(\alpha t)$ in terms of F, α and ω.

9. A multiset is a collection of objects, some of which may be repeated. So, for example, $\{1, 3, 5, 3, 2, 7, 2\}$ is a multiset. The *multiplicity* of an element in a multiset is the number of times that element occurs in the multiset. Intersections of multisets can be produced in the obvious way: given multisets A and B, the multiplicity of an element in $A \cap B$ is the minimum of its multiplicities in each of A and B.

   Given 2 multisets A amd B, with unordered elements, outline an algorithm for generating the intersection $A \cap B$.

10. Sort the following 16 numbers into 4 sets of 4 and give an explanation for each grouping: $\{1, 2, 3, 4, 4, 5, 5, 8, 9, 11, 13, 17, 25, 32, 49, 128\}$.

11. Given

$$\log_2(11) = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cdots}}}}}}}$$

   which is larger, 2^{128} or 11^{37}?

12. Write $n! = 2^a b$ where $b, n \in \mathbb{N}$ and $a \in \mathbb{N} \cup \{0\}$. Prove that $a < n$.

Longer Questions

1. Alice and Bob play Rock, Paper, Scissors until one or the other is 5 wins ahead. They generate their wins at random, so, in each round, the outcomes are equiprobably win, lose or draw.

   After 10 rounds, Alice is 1 ahead. After 13 rounds, one or the other is 1 ahead. At round 20, one of them attains the 5 win lead and the game ends. What is the probability that Alice is the ultimate winner?

2. You are monitoring a data stream which is delivering very many 32-bit quantities at a rate of 10 Megabytes per second. You know that either:

   A: All values occur equally often, *or*

   B: Half of the values occur 2^{10} times more often than others (but you dont know anything about which would be the more common values).

   You are allowed to read the values off the data stream, but you only have 2^{20} bytes of memory.

   Describe a method for determining which of the two situations, A or B, occurs. Roughly how many data values do you need to read to be confident of your result with a probability of 0.999? [This is about the 3 sigma level – 3 standard deviations of a normal distribution.]

3. A $2 \times N$ rectangle is to be tiled with 1×1 and 2×1 tiles. Prove that the number of possible tilings tends to kx^N as N gets large. Find x, to 2 decimal places.

4. The Prime Power Divisors (PPD) of a positive integer are the largest prime powers that divide it. For example, the PPD of 450 are $2, 9, 25$. Which numbers are equal to the sum of their PPD?

5. $353, 46, 618, 30, 877, 235, 98, 24, 107, 188, 673$ are successive large powers of an integer x modulo a 3-digit prime p. Find x and p.

6. $(X_1, Y_1), (X_2, Y_2), \ldots (X_n, Y_n)$ are n ordered points in the Cartesian plane that are successive vertices of a non-intersecting closed polygon.

   Describe how to find efficiently a diagonal (that is, a line joining 2 vertices) that lies entirely in the interior of the polygon.

   Repeated application of this process will completely triangulate the interior of the polygon. Estimate the worst case number of arithmetic operations needed to complete the triangulation.

7. A *semigroup* is a set S with an associative operation, which we will write as multiplication. That is, $x(yz) = (xy)z$ for all $x, y, z \in S$.

   a. Suppose that a finite semigroup S has the property that for all x there is an integer $n > 1$ such that $x^n = x$. Is it true that S is, in fact, a group?

   b. Suppose that a finite semigroup S has the property that for all x there is an integer n such that $x^{n+1} = x^n$. Show that the only subsets of S which form a group are of size 1.

8. A *finite state machine* M consists of a finite set of states, each having two arrows leading out of it, labelled 0 and 1. Each arrow from state x may lead to any state (both may lead to the same state, which might be x).

   One state is labelled the "initial" state; one or more states are labelled "accept". The machine reads a finite binary string by starting at "initial" and considering each symbol in the string in turn, moving along the arrow with that label to the next state until the end of the string is reached. If the final state is labelled "accept", the machine *accepts* the string.

   The *language*, $L(M)$ of the maching is the set of all the finite binary strings that the machine accepts.

   a. Design a machine whose language is all palindromic strings of length 6 (*i.e.*, strings for which the last 3 symbols are a reflection of the first 3).

   b. Suppose that $L(M)$ is not finite. Show that the number of strings in $L(M)$ of length n or less grows linearly with n.

   c. Show that there is no machine such that $L(M)$ consists of all twice-repeated strings.

9. Suppose P is a permutation on $\{0, 1, \ldots, n-1\}$. We want to know the length of the longest monotonically increasing subsequence. That is, the largest m such that there exist monotonically increasing $j_1, j_2, \ldots, j_m$ for which $P(j_1), P(j_2), \ldots, P(j_m)$ are also monotonically increasing.

   Describe an algorithm that will determine m using $O(n^2)$ time and $O(n)$ memory.

10. Given 541 points in the interior of a circle of unit radius, show that there must be a subset of 10 points whose diameter (the maximum distance between any pair of points) is less than $\sqrt{2}/4$.

11. Show how the edges of a cube (8 vertices; 12 edges) can be directed so that some vertices have all edges pointing out (*i.e.*, directed away from the vertex), and the remainder have one edge out and two in.

    Suppose the edges of a 5-dimensional hypercube (32 vertices, 80 edges) are directed so that all vertices have either 5 edges pointing out, or 4 edges in and 1 out. Show that every 4-dimensional subcube must contain precisely 6 of the 5-out vertices.